Math 53, Discussion 107

Wednesday, April 15, 2020 9:31 AM

16.3 (Fundamental theorem for line integrals) and 16.4 (Green's theorem)

Recall some formulae:

scalar, Param. C, r(t)

(1) $\int_C f(x,y)ds = \int_a^b f(r(t))\,\|r'(t)\|\,dt$ $\leftarrow = \sqrt{(x')^2+(y')^2}$

(2) $\int_C f(x,y)dx = \int_a^b f(r(t))\,x'(t)dt$

(3) $\int_C f(x,y)dy = \int_a^b f(r(t))\,y'(t)dt$

(4) $\int_C \langle P,Q,R\rangle \cdot dr = \int_C Pdx + Qdy + Rdz$

$\vec{F}$ vector field

f S.F

F V.F

Example: compute $\int_C xyzds$ with $r(t) = \langle 2\sin t, t, -2\cos t\rangle$, $0 \le t \le \pi$

x y z

$\|r'(t)\| = \sqrt{4\cos^2(t) + 1 + 4\sin^2(t)} = \sqrt{5}$

$\sqrt{5}\int_0^\pi (2\sin(t))(t)(-2\cos(t))\,dt$

Theorem: (Fundamental thm of line integrals)

$\int_C \nabla f \cdot dr = f(b) - f(a)$

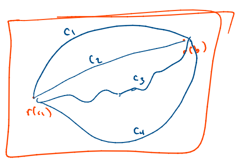
(f_x, f_y, f_z) = a vector field

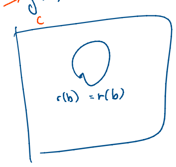
$r(t)$, $a \le t \le b$

in 1D

$\int_a^b \frac{d}{dx} f(x)dx = f(b) - f(a)$

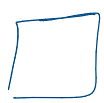
$\oint_C \nabla f = 0$





$F = Pi + Qj$ $\Longrightarrow$

Theorem: if $F = \langle P,Q\rangle$ is conservative, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ (coverse true if in a simply connected region) $\Longleftarrow$



no holes

if $F = (6x - 3y, 4x + 7y)$ is this conserv?

conservative if there exists f s.t. $f_x = 6x - 3y$

$f_y = 4x + 7y$

don't exist

if F were conservative

$\Rightarrow P_y = Q_x$. $P_y = -3$, $Q_x = 4$ ≠

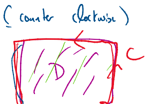
$\int_C F \cdot dr =$

Theorem: $\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

(Green's theorem)

C closed loop

Positively oriented (counter clockwise)



Exercises:

16.3

1. Compute $\int_C F \cdot dr$ for $F = \langle y^2\cos z, 2xy\cos z, -xy^2\sin z\rangle$ for $C = \langle t^2, t+1, 2t-1\rangle$ for $0 \le t \le 1$ (hint: this is conservative)
2. Determine if the following vector fields are conservative (if so, find f such that $\nabla f = F$
   $\langle 2x - 3y, -3x + 4y - 8\rangle = F$ find f s.t. $f_x = 2x - 3y$ $f_y = -3x + 4y - 8$
   $\langle e^x\cos y, e^x\sin y\rangle = F$

16.4

3. Evaluate the integral directly, then with Green's theorem $\oint_C (x - y)dx + (x + y)dy$ with C the circle at the origin and radius 2
4. Evaluate the integral using Green's theorem: $\int_C \cos y\,dx + x^2\sin y\,dy$ where C is the rectangle with vertices (0,0), (5,0), (5,2), (0,2)

$\int_x^{3x} f'(t)\,dt$

$\frac{d}{dx}$: —

$\int$:

∇ : Scalar function $\longrightarrow$ Vector field

anti: ∇^{-1} : vector field $\longrightarrow$ Scalar function

$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

$\frac{d}{dx}\int_{P(x)}^{Q(x)} f(t)\,dt$

let F be anti deriv of f

$\frac{d}{dx}\left(F(Q(x)) - F(P(x))\right)$

$= F'(Q(x))\,Q'(x) - F'(P(x))\,P'(x)$

$= f(Q(x))\,Q'(x) - f(P(x))\,P'(x)$

$P = x - y$ $\quad Q_x = 1$ $\quad Q_x - P_y = 1 - (-1) = 2$

$Q = x + y$ $\quad P_y = -1$

3) $= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_{\text{circle}} 2 \, dx\,dy = 2(4\pi) = 8\pi$

$f_x = y^2 \cos(z) \rightarrow f(x,y,z) = \int -) = xy^2 \cos(z) + \underbrace{g(y,z)}_{?}$

2) a) Conservative ✓

$f = x^2 - 3xy + 2y^2 - 8y$

b) not conservative.

FT (L) I

1) $P_{yz} = Q_{xz} = R_{xy}$

$f = xy^2 \cos(z)$

$f_x = y^2 \cos z$

$f_y = 2xy \cos(z)$

$f_z = -xy^2 \sin(z)$

$\boxed{\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(r(b)) - f(r(a))}$

$a = 0$

$b = 1$

$r(a) = r(0) = \langle t^2, t+1, 2t-1 \rangle \Big|_{t=0} = (0, 1, -1)$

$r(b) = r(1) = (1, 2, 1)$

$= 4\cos(1)$